Pulse Oximetry

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1 Oxygen content

The circulatory system distributes oxygen (O_2) throughout the body. The amount of O_2 at any given point is measured by the O_2 content (Co_{O_2}), usually given in $\frac{\text{mL } O_2 \text{at BTP}}{\text{dL blood}}$ (BTP is my acronym for body temperature and pressure). Most transported O_2 is bound to hemoglobin (Hb), but there is also free O_2 disolved directly in the plasma and cytoplasm of suspended cells.

$$Co_{O_2} = a\,[\text{Hb}]\,S_{O_2} + bP_{O_2}\ , \tag{1}$$

where $S_{O_2}$is the Hb's O_2 saturation and $P_{O_2}$is the O_2 partial pressure. Don't worry about the coefficients a and b yet, we'll get back to them in a second.

The amound of dissolved $O_2$is given by its partial pressure (P_{O_2}). Partial pressures are generally given in mm of mercury (Hg) at standard temperature and pressure (STP). Because the partial pressure changes as blood flows through the body, an additional specifier i may be added (Pi_{O_2}) to clarify the measurement location.

i	Full symbol	Location descriptor
a	Pa_{O_2}	arterial
p	Pp_{O_2}	peripheral or pulsatile
t	Pt_{O_2}	tissue
v	Pt_{O_2}	venous

O_2 is carried in the blood primarily through binding to hemoglobin monomers (Hb), with each monomer potentially binding a single O_2. Oxygen saturation (S_{O_2}) is the fraction of hemoglobin monomers (Hb) that have bound an oxygen molecule (O_2).

$$S_{O_2} = \frac{[HbO_2]}{[\text{Hb}]}\ . \tag{2}$$

S_{O_2}, as a ratio of concentrations, is unitless. It is often expressed as a percentage. [Hb] is often given in g/dL. As with partial pressures, an additional specifier i may be added (Si_{O_2}) to clarify the measurement location (Sa_{O_2}, Sp_{O_2}, ...).

Now we can take a second look at our O_2 content formula (Eq. 1). The coefficient a must convert g/dL to $\frac{\text{mL } O_2 \text{ at BTP}}{\text{dL blood}}$. Using the molecular weight of Hb and the volume of a mole of ideal gas at STP.

$$[\text{Hb}] = \chi \frac{\text{g Hb}}{\text{dL}} \cdot \frac{1 \text{ mol Hb}}{17 \text{ kg Hb}} \cdot \frac{1 \text{ mol } O_2}{1 \text{ mol Hb}} \cdot \frac{22.4 \text{ L ideal gas}}{1 \text{ mol ideal gas}} \tag{3}$$

$$= 1.32 \frac{\text{mL } O_2}{\text{g Hb}} \cdot \chi \tag{4}$$

where χ is a pure number. Therefore, $a = 1.32 \frac{\text{mL } O_2}{\text{g Hb}}$. The powers that be seem to have used a slightly different STP, since the more commonly used value is 5% higher at 1.39.

The coefficient b must convert mm Hg at STP to $\frac{\text{mL } O_2 \text{ at BTP}}{\text{dL blood}}$. Empirical experiments (?) give a value of $b = 0.003 \frac{\text{mL } O_2 \text{ at BTP}}{\text{dL blood} \cdot \text{mm Hg at STP}}$. Now we can write out the familiar form

$$Co_{O_2} = 1.39 \frac{\text{mL } O_2}{\text{g Hb}} [\text{Hb}]\, S_{O_2} + 0.003 \frac{\text{mL } O_2 \text{ at BTP}}{\text{dL blood} \cdot \text{mm Hg at STP}} P_{O_2}\ . \tag{5}$$

Reasonable levels are

[Hb]	$14 \frac{\text{g Hb}}{\text{dL blood}}$
S_{O_2}	98%
P_{O_2}	100 mm Hg at STP
$1.39 \frac{\text{mL } O_2}{\text{g Hb}} [\text{Hb}]\, S_{O_2}$	$19.1 \frac{\text{mL } O_2 \text{ at BTP}}{\text{dL blood}}$
$0.003 \frac{\text{mL } O_2 \text{ at BTP}}{\text{dL blood} \cdot \text{mm Hg at STP}} P_{O_2}$	$0.299 \frac{\text{mL } O_2 \text{ at BTP}}{\text{dL blood}}$

Because the dissolved O_2 has such a tiny contribution (1.5% of the total in my example), it is often measured at BTP rather than STP. Sometimes it is dropped from the calculation entirely. We focus on the more imporant $[\mathrm{Hb}]\,\mathrm{S}_{\mathrm{O}_2}$ in the next section.

2 Oxygen saturation

The preceding discussion used $[\mathrm{Hb}]\,\mathrm{S}_{\mathrm{O}_2}$ to represent the concentration of HbO_2 complexes (Eqs. 2 and 5). This was useful while we were getting our bearings, but now we will replace that term with a more detailed model. Let us sort the Hb monomers into species

Hb	all hemoglobin monomers
HbO_2	monomers complexed with O_2
HHb	reduced Hb (not complexed with O_2)
dysHb	dys-hemoglobin (cannot complex with O_2)
MHb	methemoglobin
HbCO	carboxyhemoglobin

These species are related as follows

$$[\mathrm{Hb}] = [\mathrm{HbO_2}] + [\mathrm{HHb}] + [\mathrm{dysHb}] \tag{6}$$

$$[\mathrm{dysHb}] = [\mathrm{MHb}] + [\mathrm{HbCO}] + \text{other broken forms} \tag{7}$$

Because modern two-color pulse-oximeters don't measure $\mathrm{S}_{\mathrm{O}_2}$exactly, the related quantity that they *do* measure has been given a name of its own: the *functional* saturation ($\mathrm{S}_{f\,\mathrm{O}_2}$).

$$\mathrm{S}_{f\,\mathrm{O}_2} = \frac{[\mathrm{HbO_2}]}{[\mathrm{HbO_2}] + [\mathrm{HHb}]}\ . \tag{8}$$

Rephrasing our earier saturation in Eq. 2, we see

$$\mathrm{S}_{\mathrm{O}_2} = \frac{[\mathrm{HbO_2}]}{[\mathrm{Hb}]} = \frac{[\mathrm{HbO_2}]}{[\mathrm{HbO_2}] + [\mathrm{HHb}] + [\mathrm{dysHb}]}\ . \tag{9}$$

To avoid confusion with $\mathrm{S}_{f\,\mathrm{O}_2}$, our original $\mathrm{S}_{\mathrm{O}_2}$ is sometimes referred to as the*fractional* saturation.

3 The Beer-Labmert law

So far we've been labeling and defining attributes of the blood. The point of this excercise is to understand how a pulse oximeter measures them. People have known for a while that different hemoglobin complexes (HbØ, HHb, MHb, HbCO, ...) have differnt absorbtion spectra (Fig. 1), and they have been using this difference since the 1930's to make pulse-oximeters based on two-color transmittance measurements[1].

By passing different wavelengths of light through perfused tissue, we can measure the relative quantities of the different Hb species. The basis for this analysis comes from the Beer-Lambert law

$$I = I_0 e^{-c\epsilon L}\ , \tag{10}$$

where I_0 is the incident intensity (entering the tissue), I is the tranmitted intensity (leaving the tissue), c is the tissue density (concentration), ϵ is the extinction coefficient (molar absorbtivity), and L is the tissue thickness. Rephrasing the math as English, this means that the intensity drops off exponentially as you pass through the tissue, and more tissue (higher c or L) or more opaque tissue (higher ϵ) mean you'll get less light out the far side. This is a very simple law, and the price of the simplicity is that it brushes all sorts of things under the rug. Still, it will help give us a basic idea of what is going on in a pulse-oximeter.

Rather than treat the the tissue as a single substance, lets use the Beer-Labmert law on a mixture of substances with concentrations c_1, c_2, ... and extinction coefficients ϵ_1, ϵ_2, ...

$$I = I_0 e^{-(c_1\epsilon_1 + c_2\epsilon_2 + \ldots)L}\ . \tag{11}$$

We also notice that the intensities and extinction coefficients may all depend on the wavelength of light λ, so we should really write

$$I_\lambda = I_{0\lambda} e^{-(c_1\epsilon_{1\lambda} + c_2\epsilon_{2\lambda} + \ldots)L}\ . \tag{12}$$

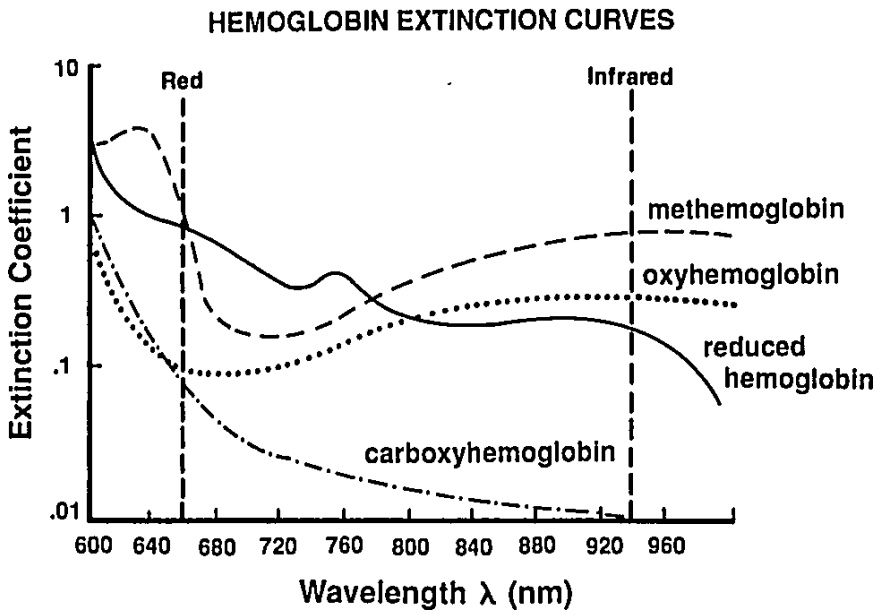


Figure 1: Absorbance spectra for assorted hemoglobin species[1].

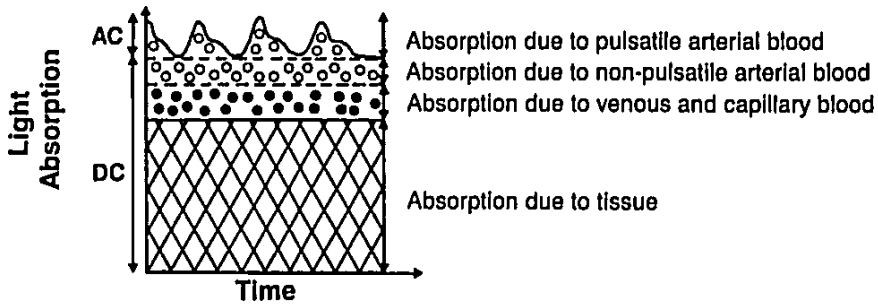


Figure 2: AC and DC transmission components[1].

Once isolated, a simple spectroscopy experiment can measure the extinction coefficient $\epsilon_{i\lambda}$ of a given species across a range of λ, and this has been done for all of our common Hb flavors. We need to play with Eq. 12 to find a way to extract the unknown concentrations, which we can then use to calculate the S_{O_2} (Eqs. 8 and 9) which we can use in turn to calculate Co_{O_2} (Eq. 5).

Note that by increasing the number of LEDs (adding new λ) we increase the number of constraints on the unknown c_i. A traditional pulse-oximeter uses two LEDs, at 660 nm and 940 nm, to measure $S_{f\,O_2}$(related to [HbO_2] and [HHb]). More recent designs called pulse CO-oximeters use more wavelengths to allow measurement of quanties related to additional species (approaching the end goal of measuring S_{O_2}).

Let us deal with the fact that there is a lot of stuff absorbing light that is not arterial blood (e.g. venous blood, other tissue, bone, etc). The good thing about this stuff is that it's just sitting there or moving through in a smooth fasion. Arterial blood is the only thing that's pulsing (Fig. 2).

During a pulse, the pressure in the finger increases and non-arterial tissue is compressed, changing L and c_i from their trough values to peak values L' and c_i'. Since the finger is big, the fractional change in width $\mathrm{d}L/L = (L' - L)/L$ is very small. Assuming the change in concentration is even smaller (since most liquids are fairly incompressible), we have

$$\frac{\mathrm{d}I_\lambda}{\mathrm{d}L} = \frac{\mathrm{d}}{\mathrm{d}L}\left(I_{0\lambda}e^{-(c_1\epsilon_{1\lambda}+c_2\epsilon_{2\lambda}+\ldots)L}\right) = \frac{\mathrm{d}}{\mathrm{d}L}\left(I_{0\lambda}e^{-XL}\right) = -XI_{0\lambda}e^{-XL} = -XI_\lambda \tag{13}$$

$$\frac{\mathrm{d}I_\lambda}{I_\lambda} = -X\mathrm{d}L\,, \tag{14}$$

where $X = c_1\epsilon_{1\lambda} + c_2\epsilon_{2\lambda} + \ldots$ is just a placeholder to reduce clutter. $\mathrm{d}I_\lambda$is the AC amplitude (height of wiggle top of the

detected light intensity due to pulsatile arterial blood), while $I_\lambda$is the DC ampltude (height of the static base of the detected light intensity due to everything else). This is actually a fairly sneaky step, because if we can also use it to drop the DC compents. Because we've assumed fixed concentrations (incompressible fluids), and there is no more DC material comming in during a pulse (by definition), we can pull out the effective L for the DC components does not change. Separating the DC and AC components and running through the derivative again, we have

$$\frac{\mathrm{d}I_\lambda}{\mathrm{d}L} = \frac{\mathrm{d}}{\mathrm{d}L}\left(I_{0\lambda}e^{-(c_{DC1\lambda}\epsilon_{DC1\lambda}+c_{DC2\lambda}\epsilon_{DC2\lambda}+\ldots)L_{\mathrm{DC}}-(c_{AC1}\epsilon_{AC1\lambda}+c_{AC2}\epsilon_{AC2\lambda}+\ldots)L}\right) \tag{15}$$

$$= I_{0\lambda}e^{-(c_{DC1\lambda}\epsilon_{DC1\lambda}+c_{DC2\lambda}\epsilon_{DC2\lambda}+\ldots)L_{\mathrm{DC}}}\frac{\mathrm{d}}{\mathrm{d}L}\left(e^{-(c_{AC1}\epsilon_{AC1\lambda}+c_{AC2}\epsilon_{AC2\lambda}+\ldots)L}\right) \tag{16}$$

$$= I_{0\lambda}Y\frac{\mathrm{d}}{\mathrm{d}L}\left(e^{-ZL}\right) = -ZI_{0\lambda}Ye^{-ZL} = -ZI_\lambda \tag{17}$$

$$\frac{\mathrm{d}I_\lambda}{I_\lambda} = -Z\mathrm{d}L\ , \tag{18}$$

where $Y = e^{-(c_{DC1\lambda}\epsilon_{DC1\lambda}+c_{DC2\lambda}\epsilon_{DC2\lambda}+\ldots)L_{\mathrm{DC}}}$ and $Z = c_{AC1}\epsilon_{AC1\lambda} + c_{AC2}\epsilon_{AC2\lambda} + \ldots$ are just placeholders to reduce clutter. Note that Eq. 18 looks just like Eq. 14 with the translation $X \to Z$. This means that if we stick to using the AC-DC intensity ratio ($\frac{\mathrm{d}I_\lambda}{I_\lambda}$) we can forget about the DC contribution completely[1].

Taking a ratio of these amplitudes at two different wavelengths, we get optical density ratio

$$R = \frac{\frac{AC_{660}}{DC_{660}}}{\frac{AC_{940}}{DC_{940}}} = \frac{\frac{\mathrm{d}I_{660}}{I_{660}}}{\frac{\mathrm{d}I_{940}}{I_{940}}} = \frac{-Z_{660}\mathrm{d}L}{-Z_{940}\mathrm{d}L} = \frac{Z_{660}}{Z_{940}}\ , \tag{19}$$

because $\mathrm{d}L$ (the amount of finger expansion during a pulse) obviously doesn't depend on the color light you are using ;). Plugging back in for Z,

$$R = \frac{c_{AC1}\epsilon_{AC1,660} + c_{AC2}\epsilon_{AC2,940} + \ldots}{c_{AC1}\epsilon_{AC1,940} + c_{AC2}\epsilon_{AC2,940} + \ldots}\ . \tag{20}$$

Assuming, for now, that there are only two species of Hb, $HbO_2$and HHb, we can solve for c_{AC1}/c_{AC2}.

$$R = \frac{c_{AC1}\epsilon_{AC1,660} + c_{AC2}\epsilon_{AC2,660}}{c_{AC1}\epsilon_{AC1,940} + c_{AC2}\epsilon_{AC2,940}} \tag{21}$$

$$R(c_{AC1}\epsilon_{AC1,940} + c_{AC2}\epsilon_{AC2,940}) = c_{AC1}\epsilon_{AC1,660} + c_{AC2}\epsilon_{AC2,660} \tag{22}$$

$$c_{AC1}(R\epsilon_{AC1,940} - \epsilon_{AC1,660}) = c_{AC2}(\epsilon_{AC2,660} - R\epsilon_{AC2,940}) \tag{23}$$

$$\frac{c_{AC1}}{c_{AC2}} = \frac{\epsilon_{AC2,660} - R\epsilon_{AC2,940}}{R\epsilon_{AC1,940} - \epsilon_{AC1,660}}\ . \tag{24}$$

So now we know $[HbO_2]\,/\,[HHb]$ in terms of the measured quantity R.

Plugging in to Eq. 8 to find the functional saturation

$$\mathrm{S}_{f\,\mathrm{O}_2} = \frac{[HbO_2]}{[HbO_2] + [HHb]} = \frac{1}{1 + \frac{[HHb]}{[HbO_2]}} = \frac{1}{1 + \frac{c_{AC2}}{c_{AC1}}} = \frac{1}{1 + \frac{R\epsilon_{AC1,940} - \epsilon_{AC1,660}}{\epsilon_{AC2,660} - R\epsilon_{AC2,940}}}\ . \tag{25}$$

As a check, we can rephrase this as

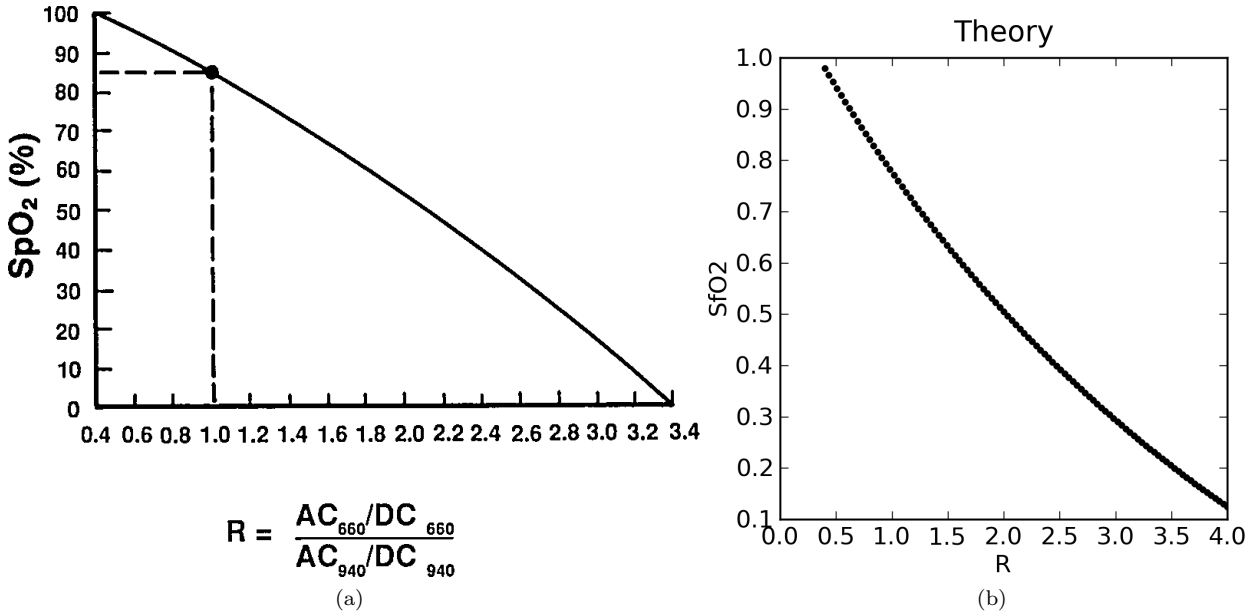
$$\mathrm{S}_{f\,\mathrm{O}_2} = \frac{1}{1 + \frac{R\epsilon_{AC1,940} - \epsilon_{AC1,660}}{\epsilon_{AC2,660} - R\epsilon_{AC2,940}}} = \frac{\epsilon_{AC2,660} - R\epsilon_{AC2,940}}{\epsilon_{AC2,660} - R\epsilon_{AC2,940} + R\epsilon_{AC1,940} - \epsilon_{AC1,660}} \tag{26}$$

$$= \frac{\epsilon_{AC2,660} - \epsilon_{AC2,940}R}{\epsilon_{AC2,660} - \epsilon_{AC1,660} + (\epsilon_{AC1,940} - \epsilon_{AC2,940})R} = \frac{-\epsilon_{AC2,660} + \epsilon_{AC2,940}R}{\epsilon_{AC1,660} - \epsilon_{AC2,660} + (\epsilon_{AC2,940} - \epsilon_{AC1,940})R}\ , \tag{27}$$

which matches Mendelson and Kent[2], Eq. 8 with the translations $\mathrm{S}_{f\,\mathrm{O}_2} \to \mathrm{Sp}_{\mathrm{O}_2}$, $R \to R/IR$, $\epsilon_{AC2,660} \to \epsilon_R(\mathrm{HHb})$, $\epsilon_{AC2,940} \to \epsilon_I R(\mathrm{HHb})$, $\epsilon_{AC1,660} \to \epsilon_R(HbO_2)$, and $\epsilon_{AC1,940} \to \epsilon_I R(HbO_2)$.

And that is the first-order explaination of how a pulse-oximeter measures the functional saturation!

[1] If the changing-L-but-static-L_{DC} thing bothers you, you can imagine insteadthat L_{DC} grows with L, but $c_{DCi\lambda}$shrinks proportially (to conserve mass). With this proportionate stretching, there is still no change in absorbtion for that component so $\frac{\mathrm{d}L}{\mathrm{d}}\exp(-c_{DCi\lambda}\epsilon_{DCi\lambda}L) = 0$ and we can still pull the DC terms out of the integral as we did for Eq. 18.



(a) (b)

Figure 3: Comparison of (a) an experimental calibration curve[1] with (b) a theoretical calibration curve calculated using Eq. 25. This is why it's a good idea to use an empirical calibration curve ;).

Reading extinction coefficients off Fig. 1, I get

$$\epsilon_{\mathrm{HbO_2},660} = \epsilon_{AC1,660} = 0.10 \tag{28}$$

$$\epsilon_{\mathrm{HHb},660} = \epsilon_{AC2,660} = 0.83 \tag{29}$$

$$\epsilon_{\mathrm{HbO_2},940} = \epsilon_{AC1,940} = 0.29 \tag{30}$$

$$\epsilon_{\mathrm{HHb},940} = \epsilon_{AC2,940} = 0.17 \tag{31}$$

which are comfortingly close to those given by Mendelson and Kent[2] in their Table 1. The corresponding $\mathrm{S}_{f\,\mathrm{O_2}}(R)$ plot can be seen in Fig. 3b.

Nomenclature

c_i	Concentration of species i
L	Length of tissue through which light must pass
R	Optical density ratio
c_{ACi}	Concentration of the ith AC species at wavelength λ
$c_{DCi\lambda}$	Concentration of the ith DC species at wavelength λ
$\mathrm{Co_{O_2}}$	Oxygen content of blood
dysHb	Dys-hemoglobin (cannot complex with $\mathrm{O_2}$)
$\epsilon_{ACi\lambda}$	Extinction coefficient of the ith AC species at waveleng th λ
$\epsilon_{DCi\lambda}$	Extinction coefficient of the ith DC species at waveleng th λ
$\epsilon_{i\lambda}$	Extinction coefficient of species i at wavelength λ
Hb	Hemoglobin monomer
HbCO	Carboxyhemoglobin

HbO_2	Hb monomers complexed with O_2
Hg	Mercury
HHb	Reduced Hb (not complexed with O_2)
I_λ	Intensity of transmitted light at wavelength λ
$I_{0\lambda}$	Intensity of incident light at wavelength λ
L_{DC}	DC finger width
MHb	Methemoglobin
O_2	Molecular oxygen
$\mathrm{P}i_{\mathrm{O}_2}$	O_2 partial pressure at location i
$\mathrm{P}_{\mathrm{O}_2}$	O_2 partial pressure
$\mathrm{S}i_{\mathrm{O}_2}$	O_2 saturation at location i
$\mathrm{S}_{\mathrm{O}_2}$	Fractional O_2 saturation
$\mathrm{S}_{f\,\mathrm{O}_2}$	Functional O_2 saturation
BTP	Body temperature and pressure
LED	Light emitting diode
STP	Standard temperature and pressure

References

[1] Kevin K. Tremper and Steven J. Barker. Pulse oximetry. *Anesthesiology*, 70(1):98–108, January 1989. ISSN 0003-3022. URL http://journals.lww.com/anesthesiology/Citation/1989/01000/Pulse_Oximetry.19.aspx.

[2] Yitzhak Mendelson and Joel C. Kent. Variations in optical absorbtion spectra of aduly and fetal hemoglobins and its effect on pulse oximetry. *IEE Transactions on Biomedical Engineering*, 36(8):844–848, August 1989. ISSN 0018-9294. URL http://ieeexplore.ieee.org/xpl/freeabs_all.jsp?arnumber=30810.